

## SPECTRALITY OF ORDINARY DIFFERENTIAL OPERATORS

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*To my parents Sarah and Moisei*

ABSTRACT. We prove the long standing conjecture in the theory of two-point boundary value problems that completeness and Dunford's spectrality imply Birkhoff regularity. In addition we establish the even order part of S.G.Krein's conjecture that dissipative differential operators are Birkhoff-regular and give sharp estimate of the norms of spectral projectors in the odd case.

Considerations are based on a new direct method, exploiting *almost orthogonality* of Birkhoff's solutions of the equation  $l(y) = \lambda y$ . This property was discovered earlier by the author.

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## 0. INTRODUCTION

**0.1. Dunford's spectrality.** Question of unconditional convergence of spectral decompositions plays a central role in the spectral theory. It is also known under other frameworks: similarity of a linear operator to a normal one, N.Dunford's spectrality, free interpolation problem [40, appendix A.3], [3, 37, 39]. In this paper we deal with complete operators with compact resolvent. For them the questions above translate respectively into unconditional basicity (UB) of eigenfunctions (ef), of root subspaces and of eigen and associated functions (eaf). The latter is the most general setting. So we shall refer to it as the (UB) problem, though sometimes will use instead the term spectrality as is common in the theory of boundary value problems (briefly bvps).

Set  $D = -id/dx$  and consider a boundary value problem (bvp) in  $L^2(0, 1)$ , defined by a differential expression

$$l(y) \equiv D^n y + \sum_{k=0}^{n-2} p_k(x) D^k y = \lambda y, \quad 0 \leq x \leq 1, \quad p_k \in L(0, 1) \quad (0.1)$$

and  $n$  linearly independent boundary conditions

$$U_j(y) \equiv \sum_{k=0}^{n-1} (a_{jk} D^k y(0) + b_{jk} D^k y(1)) = 0, \quad j = 0, \dots, n-1. \quad (0.2)$$

Spectral theory of the operator  $L$ , defined by this bvp, is thoroughly investigated during the last hundred years. The bibliography is enormous and we shall refer the reader to the fundamental monographs of M.A.Naimark [35, 36] and N.Dunford, J.T.Schwartz [3].

Remind that inverse of  $L$  is a finite-dimensional perturbation of a Volterra operator

$$L^{-1}f = Vf + \sum_{j=0}^{n-1} (f, h_j) g_j \quad (0.3)$$

where  $Vf$  gives solution to the Cauchy problem for  $l(y)$  and zero boundary conditions

$$D^j y(0) = 0, \quad j = 0, \dots, n-1. \quad (0.4)$$

Spectral investigation of this class of linear operators was initiated by A.P.Hromov [12]. During the last 20 years question of similarity to normal for operators (0.3) with dissipative  $V$  was deeply explored in works of G.M.Gubreev [10, 11] and we will dwell on it further.

The aim of this paper is three-fold. First, to give a final solution to the (UB) problem for two-point bvps (0.1)-(0.2). Second, explain why their spectrality don't fit into all existing schemes. For the reader's convenience we will show this on simplest examples. As a byproduct we give an account of relevant abstract results as well as those obtained in the classical spectral theory of bvps.

And the last but not least, we believe that solution of the (UB) problem for (0.1)-(0.2) will help in investigation of much more difficult class (0.3) with non-dissipative Volterra operator  $V$ .

**0.2. Paper outline.** In the section 1 we describe results on basisness of exponentials and ef of first and higher order bvps. It is continued by an account of the Stone-regularity in the section 3 and abstract approaches in the section 5. We consider all methods from the viewpoint of bvps, trying to reveal obstacles that prevent their usage for solving the (UB) problem. Add that modern *projection method* grew from the theory of exponentials. Obviously the latter is still important as a source of ideas, technique and inspiration for further investigations.

Background for spectral theory of bvps is exposed in the section 2. Note that it includes a new notion of a modified characteristic matrix and a new definition of regularity determinants.

Section 4 describes main results of the paper together with some open conjectures. In addition, we derive partial solution of the S.G.Krein's conjecture about spectrality of dissipative differential operators directly from one of our main results, theorem 4.3.

Proofs are placed in sections 6-8. In the section 6 we establish properties of the modified characteristic matrix. Here we deduce theorem 4.3, which ties together minimal resolvent's growth with nonvanishing of regularity determinants. Then in the section 7 we establish density properties of eigenvalues, lying in a sector off the real axis. These results form a foundation of the proof of the main theorem 4.1 in the section 8.

**0.3. Notations.** Throughout the paper components of matrices and vectors are enumerated beginning from zero. Matrices are written in boldface together with their brackets to distinguish such bracket from Birkhoff's symbol, e.g.

$$\Delta = [\Delta_{jk}]_{j,k=0}^{n-1}.$$

Different constants are denoted  $C, C_1, c$  and so on. They may vary even during a single computation. Other notations and abbreviations:

- $[a] := a + O(1/\varrho)$  stands for the Birkhoff's symbol;
- fss - fundamental system of solutions;
- $\mathbb{C}_\pm$  - upper/lower half-plane,  $\mathbb{R}$  - real axis;
- $H_\pm^2$  - Hardy space in  $\mathbb{C}_\pm$ ;
- ev - eigenvalue(s);

- cv - characteristic value(s);
- efet - entire function of exponential type;
- $\left| \Delta \xleftarrow[k]{d} \right|$  stands for determinant  $\Delta$  with the  $k$ -th column replaced by a vector  $d$ ;
- $A \asymp B$  means a double-sided estimate  $C_1 \cdot |A| \leq |B| \leq C_2 \cdot |A|$  with some absolute constants  $C_{1,2}$ , which don't depend on the variables  $A$  and  $B$ .

## 1. EXPONENTIALS AND BVPS.

**1.1. First order bvp.** Note that for  $l(y) \equiv Dy$  and general functional  $U(y)$  in the boundary condition we arrive at the classical question of unconditional basicity of exponentials. The first, now classical results in this direction were initiated by Paley and Wiener [41]. They discovered that harmonic frequencies  $k$  in the orthogonal system  $\{\exp(ikx)\}_{-\infty}^{\infty}$  may be replaced by *close* real ones  $\lambda_k$ , preserving  $\{\exp(i\lambda_k x)\}_{-\infty}^{\infty} \in (UB)$  in  $L^2(0, 2\pi)$ .

At the beginning of 1960ies B.Ya.Levin and V.D.Golovin established basis properties of exponentials whose generating function is of sine type [19, 6]. In particular, this method implies unconditional basisness with parentheses (UBP) of the first-order bvp

$$Ly = Dy \tag{1.1}$$

$$U(y) = \int_0^a y(x) d\sigma(x) \tag{1.2}$$

with finite measure  $d\sigma$  on  $[0, 1]$ , provided that

$$\sigma\{0\} \neq 0, \quad \sigma\{a\} \neq 0. \tag{1.3}$$

The ef are exponentials  $e_{kj} = x^j \exp(i\lambda_k x)$ ,  $\lambda_k$  are ev, and we meet here an example of a bvp with rather general functional in the boundary condition. Afterward the notion of a sine-type function led to one of descriptions of Riesz bases from exponentials [2]. Remind that Riesz basis is an unconditional almost normalized basis, i.e.  $C_1 \leq \|e_{kj}\|_{L^2(0,a)} \leq C_2$ . Therefore necessarily frequencies lie in a strip  $|\Im \lambda_k| \leq C$ . This implicit requirement is removed when one passes to unconditional bases from exponentials.

**1.1.1. Discovery of projection method.** However, already in 1973 B.S.Pavlov devised a simple geometric approach to this problem [42], using functional model of dissipative operators. For differential operator in a finite interval his method requires that Fourier transform of a functional in the boundary condition be a sine-type function. This requirement seems difficult to verify.

Therefore B.S.Pavlov performed a deep investigation of the asymptotic behavior of this Fourier transform and of the ev in case of the boundary condition (1.2), assuming piecewise-absolute continuity of the measure  $d\sigma$ . As a result he obtained an *effective* criterion of Riesz basisness of the root vectors in their span [42, Theorem 5]. It means that under appropriate conditions the carlesonity of the spectrum was proved and not merely put into theorem's assumptions.

1.1.2. *Semi-bounded spectrum.* Afterward Pavlov's ideas grew to the *projection method* [43], which solved the (UB) problem for exponentials in a finite interval  $[0, a]$ , provided that the spectrum  $\Lambda = \{\lambda_k\}_1^\infty$  is semi-bounded

$$\delta = \inf \Im \Lambda > -\infty, \quad \Im \Lambda := \{\Im \lambda, \lambda \in \Lambda\}. \quad (1.4)$$

Recall the criterion [43, end of p.658], modulo N.K.Nikolskii's remark cited therein at the end of p.658, see also [38]<sup>1</sup>. Let

$$b_\lambda(z) := \frac{|\lambda^2 + 1|}{\lambda^2 + 1} \frac{z - \lambda}{z - \bar{\lambda}}, \quad \lambda \neq i; \quad b_i(z) = \frac{z - i}{z + i}$$

be the Blaschke factor. Choose some  $\eta > \delta$ . Then the criterion consists of three conditions:

$$\Lambda + i\eta \in (C), \text{ i.e. } \inf_k \prod_{\substack{j=1 \\ j \neq k}}^\infty |b_{\lambda_k + \eta}(\lambda_j + \eta)| > 0. \quad (1.5)$$

$$|\varphi(x + i\eta)|^2 \in (A_2), \quad (1.6)$$

$$\varphi(z) \text{ is efet with indicator diagram } [0, a]. \quad (1.7)$$

Here  $\varphi(z)$  denotes the *generating function* of  $\Lambda$ , i.e. entire function with zeros only in  $\Lambda$  counting multiplicities. The sum in  $\Lambda + \eta$  is element-wise. (C) and (A<sub>2</sub>) stand for Carleson and Muckenhoupt conditions, see definitions in [5]. Obviously (1.4) may be replaced by

$$\inf \Im \Lambda > 0 \quad (1.8)$$

via multiplication by  $\exp(\eta x)$  for any fixed  $\eta > \inf \Im \Lambda$ .

1.1.3. *Arbitrary spectrum.* A general criterion of (UB) of exponentials without spectrum restriction has been obtained in [27]. The formulation is essentially the same as in (1.5)-(1.7) but for a new set of frequencies, obtained by reflecting  $\lambda_k \in \mathbb{C}_-$  to the upper half-plane, while  $\lambda_k \in \mathbb{C}_+ \cup \mathbb{R}$  stay the same. Later the criterion was transferred to the (UBP) case in [30]. These works used arithmetics of coinvariant subspaces of inverse shift. Note that [27] provides also another form of the criterion via distances of an appropriate unimodular symbol similar to N.K.Nikolskii's theorem [38], and thus covers incomplete systems of exponentials with arbitrary spectrum, constituting (UB) in the span.

This situation appears in applications [54] or when studying exponential bases in scales of interpolation spaces, for instance, in Sobolev spaces [1, 14].

1.1.4. *Interpolating sequences.* Apparently the (UB) problem for exponentials may be restated as interpolation problem. Namely, by Paley-Wiener theorem Fourier transform maps  $L^2(0, a)$  to the classical Paley-Wiener space  $PW^2$  of efet with indicator diagram in  $[0, a]$ , square summable on  $\mathbb{R}$ . Thus we come to the question: when does interpolation problem  $f(\lambda_k) = a_k$  have a unique solution  $f \in PW^2$  for every data  $\{a_k\}$  satisfying

$$\sum_k |a_k|^2 / \|\exp(i\lambda_k x)\|_{L^2(0, a)}^2 < \infty.$$

Such set  $\Lambda$  is called a *complete interpolating sequence* (CIS). K.Seip and Yu.I.Lyubarskii found another proof [21] of the main theorem from [27, theorem 0.1, equivalence

<sup>1</sup>In this paper N.K.Nikolskii extended projection method to systems from values of reproducing kernels.

$A) \Leftrightarrow C)$ . They reduced the problem to the boundedness of a discrete Hilbert transform, which got the following remarkable solution:

$$|\varphi(x)/\text{dist}(x, \Lambda)|^2 \in (A_2).$$

Here we don't need to reflect frequencies into  $\mathbb{C}_+$ . However the case of incomplete sequences remained uncovered. It seems that it is related to the essence of their method, see the dichotomy conjecture in [46, p.717].

1.1.5. *Criterion for functional-measure.* Note that these theorems are stated in terms of spectrum distribution and of the behaviour of the generating function. It is quite natural from the function theory viewpoint, where the given data is  $\Lambda$ . At the same time for bvps the given data are boundary conditions, and it is needed to establish a result directly in their terms. For instance, the following theorem is valid.

**Theorem A.** *Let  $d\sigma$  be a discrete measure on  $[0, a]$ . Then for (1.1) – (1.2)  $\in$  (UBP) it is necessary and sufficient that (1.3) be fulfilled.*

Sufficiency belongs to B.Ya.Levin-V.D.Golovin, whereas necessity [31] was obtained, using the general criterion from [30]. Hence in [31] it was shown how to check carlesonity of the spectrum for such general bvp. Actually (1.3) is nothing else but its Birkhoff-regularity. Note also that M.Rubnich extended theorem A to a measure with discrete and singular continuous components [44].

1.2. **Higher order bvps.** Now let us turn to ordinary differential operators. With abuse of notations assume that boundary conditions (0.2) are normalized [45]:

$$U_j(y) \equiv b_j^0 D^j y(0) + b_j^1 D^j y(1) + \dots = 0, \quad j = 0, \dots, n-1. \quad (1.9)$$

The ellipsis takes place of lower order terms at 0 and at 1;  $b_j^0, b_j^1$  are column vectors of length  $r_j$ , where

$$0 \leq r_j \leq 2, \quad \sum_{k=0}^{n-1} r_j = n, \quad \text{rank}(b_j^0 b_j^1) = r_j. \quad (1.10)$$

Evidently  $r_j = 0$  implies absence of order  $j$  conditions. In the case  $r_j = 2$  we put

$$(b_j^0 b_j^1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Below in the subsection 2.5 we define Birkhoff-regular boundary conditions. They possess a lot of remarkable spectral properties: estimate of the Green's function, asymptotics of ev and ef, equconvergence with trigonometric Fourier series on any compact  $[a, b] \subset (0, 1)$ . Moreover recently there were established necessary and sufficient conditions for equconvergence on the whole interval  $[0, 1]$ , see [13,  $n = 2$ ] and [33, Chapter 2,  $n \geq 2$ ].

However, only in 1960ties G.M.Kesel'man [16] and V.P.Mihailov [23] proved that *strong regularity* (briefly  $L \in (SR)$ ), see definition 2.3, yields (UB) of eaf.

In definition 2.3 we call *Birkhoff* but not *strongly regular* boundary conditions *weakly regular* for evident reasons and write  $L \in (WR)$ . For them A.A.Shkalikov established unconditional basicity with parentheses (two summands in each) [49, 1979]. Obviously in this case spectrality vanishes if eaf are not paired, see examples of G.M.Kesel'man [16], P.Walker [55] and J.Locker [20]. For instance, P.Walker

$\varrho \setminus n$	$2q$	$2q+1$
$\in S_0$	$q-1$	$q$
$\in S_1$	$q-1$	$q-1$

TABLE 1. Values of  $p-1$ 

considered a second-order bvp with  $\text{ef } \sin \varrho_k x$ , such that  $\text{cv } \varrho_k := \sqrt{\lambda_k}$  are divided into two sequences:

$$2\pi k, \quad k = 1, \dots; \quad 2\pi k + o(1), \quad k = 0, \dots; \quad o(1) \longrightarrow 0, \quad k \longrightarrow \infty.$$

The eigenfunctions, corresponding to two *close*  $\varrho_k$ , have an angle tending to zero. Summarizing we have

**Theorem B** (G.M.Kesel'man, V.P.Mihailov, A.A.Shkalikov).

$$L \in (SR) \Rightarrow L \in (UB), \quad L \in (WR) \Rightarrow L \in (UBP).$$

However, further investigations failed to find even a *single* bvp with the same list of properties, if Birkhoff-regularity is violated. Moreover, off this class the resolvent admits a polynomial and even exponential growth, see [4, 35] and [33, chap.I, sec.2]. Of course, there is a natural candidate for *good* boundary conditions: the self-adjoint ones, but they *are* Birkhoff-regular [45,  $n$  even], [25,  $n$  odd].

Let us recall here that *essential non-selfadjointness* of bvps stems exactly from boundary conditions and the lower order terms in  $l(y)$  play a role of small perturbation. However most of investigations of *good* (in some sense) bvps deal with perturbations of the differential expression by some subordinated functional-differential operator, see for example [7], but not for new classes of boundary conditions, except maybe papers [56, 57, 24].

## 2. GREEN'S FUNCTION.

**2.1. Birkhoff's solutions.** Set  $\varepsilon_j = \exp(2\pi i j/n)$ ,  $\varrho = \lambda^{1/n}$ ,  $|\varrho| = |\lambda|^{1/n}$ ,

$$\arg \varrho = \arg \lambda/n, \quad 0 \leq \arg \lambda < 2\pi. \quad (2.1)$$

and define sectors

$$S_\nu = \{\varrho \mid \pi\nu/n \leq \arg \varrho < \pi(\nu+1)/n\}.$$

For  $\varrho = \lambda^{1/n}$  we have that  $\varrho \in S_0 \cup S_1$ .

Let  $R_0$  be a fixed positive number such that in every sector  $S_\nu$  there exists a fss  $\{y_j(x, \varrho)\}_{j=0}^{n-1}$  of (0.1) with an exponential asymptotics:

$$D^k y_j(x, \varrho) = (\varrho \varepsilon_j)^k \cdot \exp(i\varrho \varepsilon_j x)[1], \quad j, k = 0, \dots, n-1, \quad |\varrho| \geq R_0. \quad (2.2)$$

**2.2. Canonical fss.** Note that for a given sector  $S_\nu$  there exists a number  $p$ , such that solutions  $y_j(x, \varrho)$  decay as  $j < p$  and exponentially grow otherwise (for  $x > 0$ ), except maybe a boundary ray. Clearly  $p$  depends upon the sector's choice and values of  $p-1$  are presented in the table 1.

It will be convenient to use another fss  $\{z_k\}_{k=0}^{n-1}$  of the equation (0.1):

$$z_k(x, \varrho) := \begin{cases} y_k(x, \varrho), & k = 0, \dots, p-1, \\ y_k(x, \varrho)/\exp(i\varrho \varepsilon_k), & k = p, \dots, n-1. \end{cases} \quad (2.3)$$

This choice is natural due to the fact that

$$z_k = O(1), k = 0, \dots, n-1; \quad 0 \leq x \leq 1, \quad \varrho \in S_\nu. \quad (2.4)$$

**2.3. Particular solution.** Let  $W_j(x, \varrho)$  be the algebraic complement of the element  $D^{n-1}y_j$  in the wronskian

$$W(x, \varrho) = |D^k y_j(x, \varrho)|_{j,k=0}^{n-1}.$$

Set  $\tilde{y}_j(x, \varrho) := W_j/W$ . Calculating we find that

$$\tilde{y}_j(x, \varrho) = \frac{1}{n(\varrho \varepsilon_j)^{n-1}} \exp(-i\varrho \varepsilon_j x)[1]. \quad (2.5)$$

Introducing the kernel

$$g_0(x, \xi, \varrho) = i \cdot \begin{cases} \sum_{k=0}^{p-1} y_k(x, \varrho) \tilde{y}_k(\xi, \varrho), & x > \xi \\ - \sum_{k=p}^{n-1} y_k(x, \varrho) \tilde{y}_k(\xi, \varrho), & x < \xi \end{cases}$$

we get a particular solution  $g_0(f)$  of the equation  $l(y) = \lambda y + f$ ,

$$g_0(f) := \int_0^1 g_0(x, \xi, \varrho) f(\xi) d\xi. \quad (2.6)$$

**2.4. Formula for the Green's function.** When applying boundary conditions (1.9) to a function of  $x$  and  $\varrho$ , e.g.  $z(x, \varrho)$ , it is convenient to rewrite them in vector form:

$$V(z) := (\varrho^{-j} U_j(z))_{j=0}^{n-1}.$$

Then define

$$\begin{aligned} g(x, \xi, \varrho) &= g_0(x, \xi, \varrho) \cdot (n\varrho^{n-1})/i, \\ H(\xi, \varrho) &= V_x(g(x, \xi, \varrho)). \end{aligned}$$

Here the subscript  $x$  means that the vector boundary form  $V$  acts on the kernel  $g(x, \xi, \varrho)$  over the argument  $x$ .

Observe that success of Birkhoff-regularity leans heavily upon explicit formula for the Green's function

$$G(x, \xi, \varrho) = \frac{(-1)^n \Delta(x, \xi, \varrho)}{n\varrho^{n-1} \Delta(\varrho)}, \quad (2.7)$$

$$\Delta(\varrho) = \det \mathbf{\Delta}(\varrho) = \det [V(z_0) \dots V(z_{n-1})], \quad (2.8)$$

$$\Delta(x, \xi, \varrho) = i \cdot \begin{vmatrix} z^T & g(x, \xi, \varrho) \\ \mathbf{\Delta}(\varrho) & H(\xi, \varrho) \end{vmatrix}, \quad (2.9)$$

where  $z^T$  stands for the row

$$(z_0(x, \varrho), \dots, z_{n-1}(x, \varrho)).$$

$\Delta(\varrho)$  is referred to as the *characteristic determinant*. Its estimate from below constitutes the main ingredient of the resolvent method.



### 2.5. New regularity determinants.

**Definition 2.1.** Fix some  $\varepsilon \in (0, \pi/2n)$ . Let  $S_\nu(\varepsilon)$  be the sector

$$S_\nu(\varepsilon) = \left\{ \left| \arg \varrho - \frac{(\nu + 1/2)\pi}{n} \right| \leq \varepsilon \right\}. \quad (2.10)$$

Define the regularity determinants, corresponding to the sectors  $S_\nu$ , via the formula

$$\Theta(S_\nu) := \lim_{\varrho \rightarrow \infty} \Delta(\varrho), \quad \varrho \in S_\nu(\varepsilon). \quad (2.11)$$

Let  $q = \text{entier}(n/2)$ . Then for  $0 \leq k \leq n-1$  set

$$b^i = (b_j^i)_{j=0}^{n-1}, \quad B_k^i = \left( b_j^i \cdot \varepsilon_k^j \right)_{j=0}^{n-1}, \quad i = 0, 1. \quad (2.12)$$

It is easy to calculate the limit in (2.11) for the determinant and its matrix

$$\Theta(S_\nu) = \Theta_p(b^0, b^1), \quad (2.13)$$

$$\Theta(S_\nu) = \Theta_p(b^0, b^1) := [B_k^0, k = 0, \dots, p-1 | B_k^1, k = p, \dots, n-1]. \quad (2.14)$$

The vertical line  $|$  separates columns with superscripts 0 and 1. Recall that  $p = p(\nu)$ ,  $\nu = 0, 1$ , see table 1. From (2.13) it is clear that definition 2.1 is equivalent to the standard one [45, p.361], but seems to be more natural and useful for generalizations.

**Definition 2.2.** We shall call boundary conditions (1.9) and the corresponding operator  $L$  Birkhoff-regular and write  $L \in (R)$ , if

$$\Theta(S_0) \neq 0, \quad \Theta(S_1) \neq 0. \quad (2.15)$$

**Definition 2.3.** Birkhoff-regular bvp is **strongly regular**,  $L \in (SR)$ , if either  $n$  is odd or if it is even,  $n = 2q$ , and the second order polynomial  $F(s)$  has two simple roots, where  $F(s) = \det \mathbf{F}(s)$ ,

$$\mathbf{F}(s) := [B_0^0 + s \cdot B_0^1, B_k^0, k = 1, \dots, q-1 | s \cdot B_q^0 + B_q^1, B_k^1, k = q+1, \dots, n-1]$$

Otherwise we shall call the bvp **weakly regular** and write  $L \in (WR)$ , i.e. for classes of bvps we define  $(WR) := (R) \setminus (SR)$ .

### 2.6. Modified characteristic matrix.

2.6.1. *Preliminaries.* Set

$$u_t = \begin{cases} \tilde{y}_t(\xi, \varrho) \cdot n(\varrho \varepsilon_t)^{n-1} \cdot e^{i\varrho \varepsilon_t} & = e^{i\varrho \varepsilon_t(1-\xi)} \cdot [1], & t < p \\ \tilde{y}_t(\xi, \varrho) \cdot n(\varrho \varepsilon_t)^{n-1} & = e^{i\varrho \varepsilon_t(-\xi)} \cdot [1], & t \geq p \end{cases} \quad (2.16)$$

The following formula stems immediately from definitions of  $z_k$  and  $u_t$ :

$$g(x, \xi, \varrho) = \begin{cases} + \sum_{k < p} \varepsilon_k z_k(x, \varrho) u_k(\xi, \varrho) e^{-i\varrho \varepsilon_k}, & x > \xi, \\ - \sum_{k \geq p} \varepsilon_k z_k(x, \varrho) u_k(\xi, \varrho) e^{+i\varrho \varepsilon_k}, & x < \xi. \end{cases} \quad (2.17)$$

Introduce notation

$$\# = \#(t) = \begin{cases} 1, & t < p \\ 0, & t \geq p. \end{cases}$$

We shall omit the index if it is clear from context.

**Lemma 2.4.** *The following representation holds true*

$$\begin{aligned} H(\xi, \varrho) &= \sum_{t=0}^{n-1} (-1)^{(1-\#)} \left( V_{\#}(z_t) e^{(-1)^{\#} i \varrho \varepsilon_t} \right) \cdot \varepsilon_t u_t(\xi, \varrho) \\ &= \sum_{t=0}^{n-1} (-1)^{(1-\#)} [B_t^{\#}] \cdot \varepsilon_t u_t(\xi, \varrho). \end{aligned} \quad (2.18)$$

*Proof.* Let  $V_0$  and  $V_1$  be summands of the vector functional  $V$ , corresponding to derivatives at 0 and at 1. Applying  $V_0$  and  $V_1$  to the rhs of (2.17) over variable  $x$ , we get

$$\begin{aligned} V_{0,x}(g) &= - \sum_{t \geq p} \varepsilon_t u_t(\xi, \varrho) V_0(z_t) e^{+i \varrho \varepsilon_t} \\ &= - \sum_{t \geq p} \varepsilon_t u_t(\xi, \varrho) [B_t^0], \\ V_{1,x}(g) &= + \sum_{t < p} \varepsilon_t u_t(\xi, \varrho) V_1(z_t) e^{-i \varrho \varepsilon_t} \\ &= + \sum_{t < p} \varepsilon_t u_t(\xi, \varrho) [B_t^1]. \end{aligned}$$

Here exponentials cancel out after substituting (2.2)-(2.3).  $\square$

#### 2.6.2. Main formula.

**Lemma 2.5.**

$$G(x, \xi, \varrho) = g_0(x, \xi, \varrho) - \frac{2\pi i}{n \varrho^{n-1}} \sum_{t,k=0}^{n-1} a_{tk}(\varrho) z_k(x, \varrho) u_t(\xi, \varrho), \quad (2.19)$$

where

$$a_{tk} := \begin{cases} +\frac{\varepsilon_t}{2\pi} \cdot \left| \Delta \leftarrow_k [B_t^1] \right| / \Delta, & t < p, \\ -\frac{\varepsilon_t}{2\pi} \cdot \left| \Delta \leftarrow_k [B_t^0] \right| / \Delta, & t \geq p. \end{cases} \quad (2.20)$$

*Proof.* Denote  $\Delta_k$  the  $k$ -th column of the matrix  $\Delta$ . Below we use the standard agreement that  $\widehat{\Delta}$  means absence of the corresponding column in determinant. Expanding  $\Delta(x, \xi, \varrho)$  along the topmost row and replacing  $H$  by the sum in (2.18), we obtain

$$\begin{aligned} G(x, \xi, \varrho) &= g_0(x, \xi, \varrho) + \frac{(-1)^{n_i}}{n \varrho^{n-1} \Delta} \sum_{k=0}^{n-1} (-1)^k z_k(x, \varrho) \left| \Delta_0 \dots \widehat{\Delta_k} \dots \Delta_{n-1} H \right| \\ &= g_0(x, \xi, \varrho) + \frac{(-1)^{n_i}}{n \varrho^{n-1} \Delta} \sum_{k=0}^{n-1} (-1)^k (-1)^{n-1-k} z_k(x, \varrho) \left| \Delta \leftarrow_k H \right| \\ &= g_0(x, \xi, \varrho) + \frac{-i}{n \varrho^{n-1} \Delta} \sum_{k=0}^{n-1} z_k(x, \varrho) \sum_{t=0}^{n-1} (-1)^{(1-\#)} \left| \Delta \leftarrow_k [B_t^{\#}] \right| \\ &\quad \cdot \varepsilon_t u_t(\xi, \varrho), \end{aligned}$$

whence (2.19) follows.  $\square$

Earlier the coefficients (2.20) were introduced in [33, 34]. It would be quite natural to call the matrix

$$\mathbf{A} = \mathbf{A}(\varrho) = [a_{tk}]_{t,k=0}^{n-1} \quad (2.21)$$

a *modified characteristic matrix* (mcm) of the bvp (0.1),(1.9) because it differs from the analogous object in [36, p.135] by another choice of the fss. Namely, in Naimark's book the latter is taken analytic in  $\lambda$ .

### 3. FURTHER DEVELOPMENT.

**3.1. Stone-regularity.** In case of smooth coefficients of  $l(y)$  the determinant  $\Delta(\varrho)$  admits further terms of asymptotic expansion. Assuming that necessary amount of them is not vanishing, investigators, A.P.Hromov, H.Benzinger, W.Eberhard, G.Freiling, came to the notion of Stone-regularity, see for instance [33, chap.I,sec.2] and references therein. This approach yields spectrum asymptotics, completeness and *upper* polynomial estimate of the Green's function which is *worse* than that of the Birkhoff's case. However sharpness of this upper estimate remains open.

**3.2. Expansions of smooth functions.** Stone-regularity continues to attract a lot of attention, see the recent books of J.Locker [20] and R.Mennicken, M.Möller [22]. Moreover, A.A.Shkalikov and C.Tretter succeeded to establish unconditional convergence for Stone-regular problems in classes of sufficiently smooth functions [50, 51]. Roughly speaking, the functions' smoothness should be enough to suppress the possible growth of the resolvent.

**3.3. Classification for  $D^2$ .** For simplest two-point bvps when  $l(y) \equiv D^2$  P.Lang and J.Locker [18, p.554] carried out *a complete classification* of their spectral properties! It is based on the Plücker coordinates  $p_{ij}, i < j, i, j = 0, \dots, 3$  of the matrix of coefficients in the boundary conditions (0.2). Remind that for  $n = 2$  it is a  $2 \times 4$  matrix and  $p_{ij}$  stands for its  $2 \times 2$  minor with columns  $i, j$ . These coordinates constitute a full set of the bvp invariants and are independent up to a well-known quadratic relation:

$$p_{01}p_{23} + p_{02}p_{13} + p_{03}p_{12} = 0.$$

Further in the book [20] J.Locker performed a thorough investigation of two-point bvps for  $l(y) \equiv D^n y$ . He classified degeneracy of the polynomial coefficients by the leading exponentials in the characteristic determinant and obtained the same results as for classical Stone-regular bvps, also without any claim of non-spectrality.

Thus for higher order Stone-regular problems the sharpness of the resolvent's estimate in irregular cases has not been established. Therefore it was not proved either that they are non-spectral, not speaking of two-point bvps *without* Stone-regularity assumptions. Such operators seem merely unattainable.

Summarizing we conclude that no (UB) classification has been obtained for higher order differential operators, cf. [20, remark on the p.98].

### 4. MAIN RESULTS.

**4.1. Main theorem.** The theorem below was a widely held tacit conjecture though never formulated explicitly.

**Theorem 4.1.**  $\boxed{L \in (UB) \Rightarrow L \in (R)}.$

Theorem 4.1 together with theorem B solves the (UB) problem except for weakly regular  $L$ . In the latter case we have (UBP) but need to describe the subset of (UB) bvps. Of course there are two trivial examples:

- $L \in (R)$  such that all but a finite number of ev are multiple. Then it is enough to perform Gram-Schmidt orthogonalization process and get an unconditional basis.
- $L$  is self-adjoint operator and  $L \in (WR)$ .

**Problem 4.2.** Give necessary and sufficient conditions for  $L \in (WR) \cap (UB)$ .

**4.2. Minimal resolvent's growth.** Recall the following estimate for the resolvent

$$\|R_\lambda\| \leq \frac{C}{\text{dist}(\lambda, \Lambda)} \quad (\text{LRG})$$

which is called the *Linear Resolvent Growth* condition [17]. It holds if  $L \in (UB)$ . Then inequality

$$\|R_\lambda\| \leq C |\varrho|^{-n} \quad (4.1)$$

stems from (LRG) for  $\lambda$  such that

$$\text{dist}(\lambda, \Lambda) \geq C|\lambda|. \quad (4.2)$$

Set

$$\mathbf{D} := \text{diag}(\varepsilon_0, \dots, \varepsilon_{p-1}, -\varepsilon_p, \dots, -\varepsilon_{n-1}) / (2\pi).$$

**Theorem 4.3.** Let  $L$  be a differential operator, defined by bvp (0.1), (1.9). Fix  $\nu \in \{0, 1\}$ . Given a sequence  $\{\tau_m\}_1^\infty \subset S_\nu(\varepsilon)$ , such that (4.1) fulfills for  $\varrho = \tau_m$ , we have that

$$\begin{aligned} \exists \lim_{m \rightarrow \infty} \mathbf{\Delta}(\tau_m) &= \mathbf{\Theta}_p(b^0, b^1), \\ \exists \lim_{m \rightarrow \infty} \mathbf{A}(\tau_m) &=: \mathbf{A}_\infty(L) = \mathbf{\Theta}_p(b^0, b^1)^{-1} \cdot \mathbf{\Theta}_p(b^1, b^0) \cdot \mathbf{D} \end{aligned}$$

and all the matrices are invertible.

Therefore in order to prove theorem 4.1 it is enough to establish (4.1) for one sequence in  $S_0(\varepsilon)$ , another in  $S_1(\varepsilon)$  and apply theorem 4.3.

Converse to theorem 4.3 is also true, namely, nonvanishing of one regularity determinant yields minimal resolvent's growth in the corresponding sector of the  $\lambda$ -plane [57].

#### 4.3. Dissipative case.

**Conjecture 4.4** (S.G.Krein). <sup>2</sup> Two-point dissipative bvps are Birkhoff-regular.

**Theorem 4.5.** Even-order dissipative differential operators are Birkhoff-regular.

This theorem is an immediate corollary of the theorem 4.3. Indeed, if  $L$  is dissipative, then (LRG) is valid in the whole half-plane  $\mathbb{C}_-$ . Therefore the corresponding regularity determinant  $\Theta(S_1)$  is nonzero. It is enough in the even-order case, since then the second determinant is the same (in our definition, see (2.13) and the value of  $p - 1$  in the table 1).

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<sup>2</sup>V.A.II'in (personal communication) has kindly informed us that it was conjectured by S.G.Krein during one of Voronez mathematical schools in seventies-eighties.

*Remark 4.6.* Recently E.A.Shiryaev found another proof of theorem 4.5 [48]. He established self-adjointness of senior terms of even-order dissipative boundary conditions. In the odd-order case he also presented an example where regularity is violated [48, section 4]. This raises a difficult question which odd order dissipative boundary value problems are Birkhoff-regular (if any).

However, one nonzero determinant, namely  $\Theta(S_1)$ , is enough to assert that odd-order dissipative differential operator  $L_{odd}$  is *half-regular* in the sense of [24]. Note that the conjugate  $L_{odd}^*$  is also *half-regular* by the same reasoning because its resolvent obeys (LRG) in  $\mathbb{C}_+$ .

The following theorem sheds light on behaviour of the spectral projectors of  $L_{odd}$ .

**Theorem 4.7.** *Let  $L_{odd}$  be an odd order dissipative differential operator, generated by bvp (0.1),(1.9). Assume that  $\lambda_m$  is a simple ev (of multiplicity 1). For ef  $u_m$  of  $L_{odd}$  and the biorthogonal ef  $v_m$  of  $L_{odd}^*$  we have that  $(u_m, v_m) = 1$ . Let  $\varrho_m := \lambda_m^{1/n}$  be the corresponding cv,  $\Im \varrho_m \geq 0$ . Then the spectral projector*

$$P_m(f) = (f, v_m)u_m, \quad f \in L^2(0, 1)$$

*admits a sharp norm estimate as an operator in  $L^2(0, 1)$*

$$\|P_m\| \asymp \exp(\Im \varrho_m)/(1 + \Im \varrho_m). \quad (4.3)$$

*Proof.* Let  $\{\tilde{z}_k(x, \bar{\varrho})\}_{k=0}^{n-1}$  be the canonical fss of the conjugate equation

$$l^*(z) = \bar{\lambda}z, \quad \lambda \in \mathbb{C}_-.$$

Expanding  $v_m$  along this fss,  $u_m$  along the fss (2.3) and invoking equation (3.3) from [24], we find an asymptotic representation

$$u_m(x) = c_0 \cdot z_0(x, \varrho_m)[1], \quad (4.4)$$

$$v_m(x) = d_0 \cdot \tilde{z}_0(x, \bar{\varrho}_m)[1]. \quad (4.5)$$

Recall that

$$z_0(x, \varrho) = \exp(i\varrho x)[1],$$

$$\tilde{z}_0(x, \bar{\varrho}) = \exp(i\bar{\varrho}(x-1))[1],$$

$$\|z_0\| \asymp \|\tilde{z}_0\| \asymp (1 + |\Im \varrho|)^{-1/2}$$

$$\|P_m\| = |c_0| \cdot |d_0| \cdot \|z_0\| \cdot \|\tilde{z}_0\|.$$

Remove for simplicity brackets since they don't affect considerations. Then

$$1 = (u_m, v_m) = c_0 \cdot \bar{d}_0 \cdot \exp(i\varrho_m)$$

whence estimate (4.3) readily follows.  $\square$

## 5. ABSTRACT APPROACH.

**5.1. Functional model.** So far we examined the resolvent approach to spectrality. Now it's a time to look at the results for abstract linear operators and to try to apply them to the operator  $L$ . First, remind that the spectral theory of abstract non-selfadjoint operators has been deeply investigated, using M.S.Lifshič's characteristic function. Presently serious attempts are made to develop spectral theory for operators close to unitary [15], equivalently close to a self-adjoint operator, particularly translating all constructions to the language of differential operators [52]. However it still can not provide solution of the (UB) problem.

This approach relies on the functional model theory, which is most deeply explored for dissipative operators [47, 39]. Fundamental contribution has been done by A.S.Marcus, V.E.Katsnelson, N.K.Nikolskii, B.S.Pavlov, V.I.Vasyunin and S.R.Treil, resulting finally in a strong criterion of unconditional basicity of ef [39, Lect. VI, IX] and even of a family of invariant subspaces [53]. Let us state this remarkable result in a simple form, suitable for differential operators.

**Theorem C.** *Assume that the differential expression (0.1) is formally self-adjoint and  $L$  is a dissipative operator. Then*

$$L \in (UB) \text{ in the span of eaf} \Leftrightarrow \text{uniform minimality of eaf.}$$

In abstract situation *uniform minimality* is much weaker than (UB). Nevertheless, the former seems to be unverifiable for differential operators. The unique class of two-point bvps where it is known to be valid is (SR). However, in this case (UB) is *already established*! Hence, this approach turns out to be ineffective for bvps.

*Remark 5.1.* From theorem 4.7 stems *uniform minimality* of ef of odd order differential operator, provided that cv are simple and lie in a strip:

$$|\Im \rho_k| \leq C. \quad (5.1)$$

If we would be able to verify these assumptions, then we were able to apply theorem C. But presently neither of these conditions is possible to check.

**5.2. Gubreev's development of projection method.** Presently probably the most promising *abstract* approach is worked out by G.M.Gubreev. He succeeded to develop further Pavlov's *projection method* for finite-dimensional perturbations of Volterra dissipative operators [10, 11]. To give a taste of this approach, we'll state one of his results in the simplest situation, omitting minor details for brevity.

**5.2.1. One-dimensional perturbation of  $J_a$ .** Let  $J_a$  be an integration operator in  $L^2(0, a)$ ,

$$J_a f(x) = \int_0^x f(t) dt, \quad f \in L^2(0, a),$$

$A$  be an unbounded operator, such that  $A^{-1}$  is a one-dimensional perturbation

$$A^{-1}h = J_a h + (h, f)g.$$

Equivalently,  $\mathcal{D}_A = \ker \hat{\varphi}$  for some unbounded functional  $\hat{\varphi}$ . Then

$$\varphi(z) := \hat{\varphi}(\exp(iaz))$$

is the generating function of the spectrum  $\Lambda$  of  $A$ .

Assume that  $\Lambda$  lies strictly in  $\mathbb{C}_+$ , i.e obeys (1.8), and that ev are simple. Eigenfunctions  $g(\lambda_k)$  are values of the vector-valued entire function

$$g(z) := (I - zJ_a)^{-1}g.$$

Set  $w^2(x) := \|g(x)\|_{L^2(0, a)}^2$ . It is a weight on  $\mathbb{R}$ . Let  $w_-(z)$  be an outer function in  $\mathbb{C}_-$  such that  $|w_-(x)| = w(x)$ ,  $x \in \mathbb{R}$  [5].

**Theorem D.**  $A^{-1} \in (UB)$  iff

- (1)  $\varphi$  is efet with indicator diagram  $[0, a]$ ;
- (2)  $g$  coincides with restriction on  $(0, a)$  of some function  $g_w(x)$ ,  $x \in \mathbb{R}_+$ , see [10], generated by Muckenhoupt weight  $w^2|_{\mathbb{R}} \in (A_2)$ ;
- (3)  $|\varphi(x - i\eta) \cdot w_-(x - i\eta)|^2 \in (A_2)$ , where  $\eta > 0$  and is fixed;

(4)  $\Lambda \in (C)$ .

Actually condition 1 is completeness of  $A$ , 2 and 3 are its similarity to the model dissipative operator  $D_M := D$  in  $L^2(\mathbb{R})$  with domain

$$E = \text{span}(\{e^{i\lambda_k x}\}_{k=1}^\infty).$$

Fourier transform maps  $E$  onto the coinvariant subspace  $H(B) \subset H_+^2$ , generated by the Blaschke product  $B(z) := \prod_{\lambda \in \Lambda} b_\lambda(z)$ , while operator  $D_M$  transforms to

$$A_{\text{mod}} f(z) = z f(z) - \lim_{z \rightarrow \infty} z f(z).$$

This can be easily verified, applying  $D_M$  to the basis elements  $e^{i\lambda_k x}$  and performing Fourier transform. In other words conditions 2,3 assert that  $\hat{\varphi}$  is a right Delsarte functional on the Sobolev space  $W_2^1(0, a)$  [10, Theorem 3.3]. At last, condition 4 states that ef of  $D_M$  form an unconditional basis in their span  $E$ .

Note that for  $w(x) \equiv 1$  we have also  $g \equiv 1$ . Then  $A$  can be written as

$$A = D, \quad \mathcal{D}_A = \ker \hat{\varphi}$$

and we return back to bases from exponentials.

**5.2.2. One-dimensional perturbation of a dissipative operator.** Similar results are valid for one-dimensional perturbations  $A^{-1}f = Bf + (f, h)g$  of abstract dissipative Volterra operator  $B$  with  $(I - \lambda B)^{-1}$  being an efet. Define a vector-valued function

$$g(z) := (I - zB)^{-1}g$$

which is called a quasi-exponential. Then (UB) is established for the family

$$\{g(\lambda_k)\}_{k=1}^\infty$$

under assumption (1.4), provided that weight  $w^2 \in (A_2)$ , where  $w^2(s) := \|g(s)\|^2$  on  $\mathbb{R}$ . The necessity of the latter condition was announced in a difficult to attain article [8]. Being unaware of this fact we reproved it when imaginary part of  $B$  is finite-dimensional [32]. Soon after G.M.Gubreev removed this restriction [9]. Moreover, condition (1.4) may be weakened to the requirement that there is a horizontal strip free of spectrum [11]

$$|\Im \lambda_k| \geq h > 0. \quad (5.2)$$

Note that for exponentials such case was treated in [28] and served as a guideline for the proof of (UB) without spectrum restrictions [27].

**5.2.3. Finite-dimensional perturbation.** In the review [10] one can also find theorems, concerning (UB) for linear combinations of quasi-exponentials. It is easy to see that second order bvps lead to such systems. Say, for  $l(y) \equiv D^2$  the ef are

$$c_k g(\lambda_k) + d_k g(\lambda_k) \equiv c_k \exp(i\lambda_k x) + d_k \exp(i\lambda_k x), \quad x \in [0, 1].$$

Both of the systems  $\{g(\lambda_k)\}_1^\infty, \{g(-\lambda_k)\}_1^\infty$  are candidates for (UB) in  $L^2(0, 1)$ , whence we arrive at question of building an unconditional basis from two others. However, it is not the case when  $n > 2$ . Namely, assuming for simplicity  $l(y) \equiv D^n$ , we expand an ef into linear combinations of  $n$  exponentials

$$\exp(iz\varepsilon_j x), \quad j = 0, \dots, n-1$$

at the point  $z = \lambda_k$ . Here only one ( $n$  odd) or two ( $n$  even) of the systems  $\{\exp(i\lambda_k \varepsilon_j x)\}_{k=1}^\infty$  can be candidates for (UB). So for higher order ( $n > 2$ ) bvps the ef family no longer fits into this scheme.

Add also that spectrum properties like  $\Lambda \in (C)$  or (5.2) with  $\text{ev } \lambda_k$ , replaced by  $\text{cv } \varrho_k = \lambda_k^{1/n}$ , are difficult to translate to some restrictions, imposed on boundary conditions. Nevertheless theorem A demonstrates that for the former it is possible.

## 6. LIMIT OF MCM.

**6.1. Almost orthogonality.** An *almost orthogonality* property was discovered in [26] for ordinary differential equations. In [29] it was transferred to quasidifferential expressions with a summable coefficient by the  $(n-1)$ -st derivative. For Birkhoff's fss of the equation (0.1) with asymptotics (2.2) *almost orthogonality* asserts that

$$\left\| \sum_{k=0}^{n-1} c_k y_k(x, \varrho) \right\|_{L^2(0,1)}^2 \asymp \sum_{k=0}^{n-1} |c_k|^2 \|y_k(x, \varrho)\|_{L^2(0,1)}^2 \quad (6.1)$$

for any coefficients  $c_k$ , which may vary with  $\varrho$ .

*Remark 6.1.* Moreover, the system (2.16) is also *almost orthogonal*. This is valid because the latter has also an exponential asymptotics and this is the unique ingredient needed for this property [29].

**6.2. Boundedness of mcm.** Below  $R_0$  is the positive number from (2.2).

**Lemma 6.2.** *The integral operator  $g_0$  in  $L^2(0, 1)$ , see (2.6), admits an estimate*

$$\|g_0\| \leq C|\varrho|^{-n}, \quad \varrho \in S_\nu(\varepsilon), \quad |\varrho| \geq R_0. \quad (6.2)$$

*Proof.* Removing brackets from the asymptotic expressions for the functions  $y_k(x, \varrho)$  and  $\tilde{y}_k(\xi, \varrho)$ , we obtain a kernel  $G_0(x, \xi, \varrho)$ , which naturally extends to  $\mathbb{R}$ . So

$$g_0(x, \xi, \varrho) = G_0(x, \xi, \varrho) + O\left(\frac{1}{\varrho^n}\right).$$

Obviously, the extended kernel coincides with the Green's function of the self-adjoint operator  $D^n$  in  $L^2(\mathbb{R})$ . The latter obeys an analogue of (6.2) in  $L^2(\mathbb{R})$ . All the more an integral operator with the kernel  $G_0(x, \xi, \varrho)$  obeys the same estimate in  $L^2(0, a)$ , which completes the proof.  $\square$

**Lemma 6.3.** *Let  $P = P(\varrho)$  be a finite dimensional operator in  $L^2(0, 1)$  with the kernel*

$$P(x, \xi, \varrho) = \sum_{t,k=0}^{n-1} a_{tk}(\varrho) z_k(x, \varrho) u_t(\xi, \varrho).$$

*Then*

$$\|P\| \leq C/|\varrho|, \quad |\varrho| \geq R_0, \quad \varrho \in S_\nu(\varepsilon). \quad (6.3)$$

*In addition a double sided estimate holds*

$$\|P\| \asymp \sqrt{\sum_{t,k=0}^{n-1} |a_{tk}(\varrho)|^2} \cdot \frac{1}{|\varrho|^2}, \quad \varrho \in S_\nu(\varepsilon), \quad |\varrho| \geq R_0. \quad (6.4)$$

*Proof.* First observe that (6.3) stems readily from (4.1), (6.2) and (2.19). Further, let  $f \in L^2(0, 1)$ . Then

$$Pf = \sum_{k=0}^{n-1} d_k z_k(x, \varrho),$$



where

$$d_k = \sum_{t=0}^{n-1} a_{tk}(\varrho) \int_0^1 f(\xi) u_t(\xi, \varrho) d\xi.$$

Invoking *almost orthogonality* property (6.1), we arrive at the relation:

$$\|Pf\|_{L^2(0,1)}^2 \asymp \sum_{k=0}^{n-1} |d_k|^2 \|z_k\|_{L^2(0,1)}^2.$$

Next, a direct calculation shows that

$$\|z_k\|^2 \asymp \|u_t\|^2 \asymp \frac{1}{|\varrho|}, \quad \varrho \in S_\nu(\varepsilon), \quad |\varrho| \geq R_0. \quad (6.5)$$

Introduce a sum

$$T_k = \sum_{t=0}^{n-1} |a_{tk}(\varrho)|^2.$$

Then (6.4) reduces to

$$\sup_{\|f\| \leq 1} \sum_{k=0}^{n-1} |d_k|^2 \asymp \frac{1}{|\varrho|} \sum_{t,k=0}^{n-1} |a_{tk}(\varrho)|^2 = \frac{1}{|\varrho|} \sum_{k=0}^{n-1} T_k. \quad (6.6)$$

Fix  $\varrho$  and suppose that  $T_k$  attains its maximum for  $k = k_0(\varrho)$ . Then it suffices to check (6.6) when its rhs reduces to one summand:

$$\sup_{\|f\| \leq 1} \sum_{k=0}^{n-1} |d_k|^2 \asymp \frac{1}{|\varrho|} T_{k_0}. \quad (6.7)$$

Obviously the lhs of (6.6) has trivial bounds

$$\sup_{\|f\| \leq 1} |d_{k_0}|^2 \leq \text{lhs (6.6)} \leq \sum_{k=0}^{n-1} \sup_{\|f\| \leq 1} |d_k|^2. \quad (6.8)$$

Applying (6.5) and remark 6.1, we get that for fixed  $k$

$$\sup_{\|f\| \leq 1} |d_k|^2 = \left\| \sum_{t=0}^{n-1} a_{tk} u_t(\cdot, \varrho) \right\|_{L^2(0,1)}^2 \asymp \sum_{t=0}^{n-1} |a_{tk}|^2 \|u_t\|^2 \asymp T_k \frac{1}{|\varrho|}.$$

It allows to rewrite (6.8) as

$$\text{lhs (6.6)} \asymp T_{k_0} \frac{1}{|\varrho|}$$

which coincides with (6.7). □

*Corollary 6.4.* The mcm is bounded

$$\sum_{t,k=0}^{n-1} |a_{tk}(\varrho)|^2 = O(1), \quad \varrho \in S_\nu(\varepsilon), \quad |\varrho| \geq R_0. \quad (6.9)$$

Indeed, one should compare (6.3) and (6.4).

**6.3. Proof of theorem 4.3.** Let  $A_t$  be the  $t$ -th column of the matrix (2.21)

$$\mathbf{A} = [A_0, \dots, A_{n-1}].$$

In virtue of (2.20)  $A_t$  satisfies an equation

$$\Delta A_t = (-1)^{(1-\#)} \cdot \frac{\varepsilon_t}{2\pi} [B_t^\#], \quad 0 \leq t \leq n-1. \quad (6.10)$$

**Lemma 6.5.** *For every  $t \in \{0, \dots, n-1\}$  there exists a vector  $\eta_t \in \mathbb{C}^n$  such that*

$$\Theta_p(b^0, b^1) \eta_t = (-1)^{(1-\#)} \cdot \frac{\varepsilon_t}{2\pi} B_t^\#. \quad (6.11)$$

*Proof.* Fix  $t \in \{0, \dots, n-1\}$ . Using compactness of the set of vectors

$$A_t(\varrho), \quad \varrho \in S_\nu(\varepsilon), \quad |\varrho| \geq R_0$$

we deduce existence of a limiting vector  $\eta_t$ :

$$\eta_t = \lim_{l \rightarrow \infty} A_t(\varrho_{m_l}) \quad (6.12)$$

for some subsequence  $\varrho_{m_l}$ . In the meantime the formula

$$\lim_{l \rightarrow \infty} \Delta(\varrho_{m_l}) = \Theta_p(b^0, b^1) \quad (6.13)$$

stems directly from (2.14). Combine (6.10), (6.12) and (6.13), and we are done.  $\square$

**Lemma 6.6.** *Denote  $R(\mathbf{A})$  the image of matrix  $\mathbf{A}$ . Then*

$$R(\Theta(b^0, b^1)) \supset \text{span}(B_0^0, \dots, B_{p-1}^0, B_p^1, \dots, B_{n-1}^1). \quad (6.14)$$

$$R(\Theta(b^0, b^1)) \supset \text{span}(B_0^1, \dots, B_{p-1}^1, B_p^0, \dots, B_{n-1}^0). \quad (6.15)$$

*Proof.* First, apply the matrix  $\Theta_p(b^0, b^1)$  to the standard basis in  $\mathbb{C}^n$  and get (6.14). Second, (6.15) follows from (6.11) when  $t$  runs over  $0, \dots, n-1$ .  $\square$

**Lemma 6.7.** *The matrix  $\Theta_p(b^0, b^1)$  is invertible.*

*Proof.* Set

$$\mathbf{Q} = [\mathbf{Q}^0, \mathbf{Q}^1], \quad \mathbf{Q}^i = [B_0^i \dots B_{n-1}^i], \quad \Psi = [\varepsilon_j^k]_{j,k=0}^{n-1}.$$

Inclusions (6.14)-(6.15) yield that

$$R(\Theta_p(b^0, b^1)) \supset \text{span} \mathbf{Q}.$$

But a  $(j, k)$ th block-entry of the product  $\mathbf{Q}^i \cdot \Psi^*$  is an  $r_j \times 1$  vector

$$b_j^i \sum_{t=0}^{n-1} \varepsilon_j^t \cdot \overline{\varepsilon_k^t} = b_j^i \cdot n \cdot \delta_{jk}, \quad i = 0, 1; \quad j, k = 0, \dots, n-1,$$

whence

$$\frac{1}{n} \mathbf{Q} \Psi^* = \mathbf{B} := \begin{pmatrix} b_0^0 & & b_0^1 \\ & \ddots & \\ & & b_{n-1}^0 & & b_{n-1}^1 \end{pmatrix}. \quad (6.16)$$

Due to (1.10)  $\text{rank} \mathbf{B} = \sum_{j=0}^{n-1} r_j = n$ . Therefore

$$R(\mathbf{Q} \Psi^*) = R(\mathbf{B}) = \mathbb{C}^n.$$

Since  $\Psi$  is invertible,  $\mathbf{Q}$  is a full range matrix, and the same is  $\Theta_p(b^0, b^1)$ .  $\square$

## 7. SPARSENESS OF CV.

**7.1. Estimate off cv.** Denote  $\Gamma = \{\varrho_j\}_{j=1}^\infty$ , the sequence of all distinct cv not counting multiplicities. Fix  $\nu \in \{0, 1\}$  and let

$$\Gamma_\varepsilon := \Gamma \cap S_\nu(\varepsilon). \quad (7.1)$$

Draw a hyperbolic circle

$$K(\varrho_j, \delta) = \{\varrho : |b_{\varrho_j}(\varrho)| \leq \delta\}$$

around every  $\varrho_j \in \Gamma$ , remove them from  $S_\nu(\varepsilon)$  and denote  $S_\nu(\varepsilon, \delta)$  the remaining domain. Set

$$D(\varrho, \delta) := \{z - \varrho \leq \delta|\Im \varrho|\}.$$

Below we shall often use relations from [39, Lecture XI, formulas after (9)]

$$K(\varrho, \delta) \supset D(\varrho, \delta), \quad (7.2)$$

$$K(\varrho, \delta) \subset D(\varrho, \delta_1), \quad \delta_1 = \frac{2\delta}{1-\delta}. \quad (7.3)$$

**Lemma 7.1.** *Let  $\varrho \in S_\nu(\varepsilon, \delta)$ . Then  $|\varrho - \varrho_j| \geq c|\varrho|$ ,  $\forall \varrho_j \in \Gamma$ .*

*Proof.* Let for definiteness  $\nu = 0$ . If  $\varrho_j \in S_1$  then

$$|\varrho - \varrho_j| \geq \text{dist}(\varrho, \partial S_1) \geq |\varrho| \cdot \sin\left(\left(\frac{\pi}{n} - \left(\frac{\pi}{2n} + \varepsilon\right)\right)\right) \geq c|\varrho|.$$

If  $\varrho_j \in S_0$  then  $|b_\varrho(\varrho_j)| = |b_{\varrho_j}(\varrho)| > \delta$  according to the choice of  $\varrho$ . So  $\varrho_j \notin K_\varrho(\delta)$ . From (7.2) stems  $\varrho_j \notin D(\varrho, \delta)$ , i.e.

$$|\varrho_j - \varrho| > \delta|\Im \varrho| \geq \delta \cdot \sin\left(\frac{\pi}{2n} - \varepsilon\right)$$

where we used that  $\arg \varrho \geq \left(\frac{\pi}{2n} - \varepsilon\right)$ .  $\square$

**Lemma 7.2.** *Let  $\varrho \in S_0 \cup S_1$ ,  $\varrho_j \in \Gamma$ . Then*

$$|\varrho\varepsilon_m - \varrho_j| \geq c|\Im \varrho|, \quad m = 1, \dots, n-1.$$

*Proof.* Let for simplicity  $m = 1$ . Other  $m$  may be considered in a similar way. Then  $\varrho\varepsilon_1 \notin S_0 \cup S_1$  while  $\varrho_j$  belongs to this union. Assume  $n > 2$ . Then

$$|\varrho\varepsilon_1 - \varrho_j| \geq \text{dist}(\varrho\varepsilon_1, \partial S_1) \geq \text{dist}(\varrho\varepsilon_1, \{\arg z = \arg \varepsilon_1\}) = \text{dist}(\varrho, \mathbb{R}_+) \geq |\Im \varrho|.$$

If  $n = 2$  then  $\varrho\varepsilon_1 = -\varrho$ ,  $S_0 \cup S_1 = \mathbb{C}_+$  and  $\text{dist}(-\varrho, \mathbb{C}_+) = |\Im \varrho|$ .  $\square$

**Lemma 7.3.** *Let  $\varrho \in S_\nu(\varepsilon, \delta)$ ,  $\lambda = \varrho^n$ . Then the estimate (4.1) holds true.*

*Proof.* We use the identity

$$|\varrho^n - \varrho_j^n| = \prod_{m=0}^{n-1} |\varrho\varepsilon_m - \varrho_j|,$$

estimate its factors for  $m = 0$  via lemma 7.1, others by lemma 7.2 and thus arrive at (4.2), whence (4.1) follows.  $\square$

**7.2. ef properties.** Choose some circle  $K(\varrho, \delta)$ , fixed an integer  $N$  and take any  $N$  elements from  $\Gamma \cap K(\varrho, \delta)$ . Enumerate these cv  $\varrho_1, \dots, \varrho_N$  and denote respective ef  $u_1, \dots, u_N$ . Set

$$\omega_{lq}(x, \varrho) := \frac{1}{q!} \frac{d^q}{d\varrho^q} \omega_{l0}(x, \varrho), \quad l = 0, \dots, n-1 \quad (7.4)$$

where  $\omega_{l0}(x, \varrho) := z_l(x, \varrho)$ . We used functions  $\omega_{lq}$  extensively in [33, chap.4]. It is easy to verify the estimate [33, chap.4, lemma 5.1]

$$\|\omega_{lq}(x, \varrho)\|^2 \asymp \frac{1}{|\varrho|^{2q+1}}, \quad \varrho \in S_\nu(\varepsilon). \quad (7.5)$$

Next, expanding the ef  $u_j(x)$  over the system (2.3), we get

$$u_j(x) = \sum_{l=0}^{n-1} d_{jl} \omega_{l0}(x, \varrho_j), \quad j = 1, \dots, N. \quad (7.6)$$

Due to *almost orthogonality* of the fss (2.3)

$$\|u_j\|^2 \asymp \sum_{l=0}^{n-1} |d_{jl}|^2 \cdot \|\omega_{l0}\|^2 \asymp \sum_{l=0}^{n-1} |d_{jl}|^2 \cdot \frac{1}{|\varrho_j|}. \quad (7.7)$$

**7.3. Norm of a linear combination of ef.** Now let  $u(x)$  be some linear combination of the ef  $\{u_j(x)\}_1^N$

$$u(x) = \sum_{j=1}^N c_j \cdot u_j(x) \quad (7.8)$$

Normalize the ef  $u_j, j = 1, \dots, N$ . Since they are subsystem of an unconditional basis, the norm equivalence holds true

$$\|u\|^2 \asymp \sum_{j=1}^N |c_j|^2 \cdot \|u_j\|^2 \asymp \sum_{j=1}^N |c_j|^2. \quad (7.9)$$

**7.4. Canonical ef representation.** From the other hand, the following representation is also valid [33, chap.4]

$$u(x) = \sum_{l=0}^{n-1} \sum_{m=0}^{\infty} a_{ml} \cdot \omega_{lm}(x, \varrho) \quad (7.10)$$

where

$$a_{ml} = \sum_{j=1}^N c_j \cdot d_{jl} \cdot (\varrho_j - \varrho)^m. \quad (7.11)$$

We merely substituted (7.6) into (7.8) and expanded every summand  $\omega_{l0}(x, \varrho_j)$  into Taylor series centered in  $\varrho$ . Moreover, the following estimate is valid [33, chap.4]

$$\|u\|^2 \asymp \sum_{l=0}^{n-1} \sum_{m=0}^{N-1} |a_{ml}|^2 \cdot \|\omega_{lm}(x, \varrho)\|^2 \asymp \sum_{l=0}^{n-1} \sum_{m=0}^{N-1} |a_{ml}|^2 \cdot \frac{1}{|\varrho|^{2m+1}}, \quad \varrho \in \Gamma_\varepsilon. \quad (7.12)$$

In fact there we assumed that  $\varrho$  lies in a strip  $|\Im \varrho| \leq C$  but the case  $\varrho \in S_\nu(\varepsilon)$  may be considered along the same lines and even simpler. Note that the row  $(a_{00}, \dots, a_{0,n-1})$  coincides with linear combination of matrix  $d$  rows,

$$d = [d_{jl}]_{j=1, l=0}^{N, n-1}. \quad (7.13)$$

**7.5. Spectrum sparseness.** Recall a definition of a sparse sequence.

**Definition 7.4.** Let  $P$  be a sequence of points in  $\mathbb{C}_+$ . Then  $P \in (S)$  if for some  $\delta > 0$

$$K(\varrho, \delta) \cap K(\mu, \delta) = \emptyset, \quad \varrho \neq \mu, \quad \varrho, \mu \in P.$$

Equivalently each circle  $K(\varrho, \delta)$  contains  $\leq 1$  element from  $P$ .

**Definition 7.5.**  $P$  is an  $N$ -sparse sequence,  $P \in (NS)$  if for some  $\delta > 0$

$$\#(P \cap K(\varrho, \delta)) \leq N, \quad \forall \varrho \in S_0 \cup S_1.$$

**Lemma 7.6.** Fix  $\nu \in \{0, 1\}$  and consider the set  $\Gamma_\varepsilon$ , see (7.1). Then for sufficiently small  $\delta$  in definition 7.5  $\Gamma_\varepsilon \in (nS)$ .

*Proof.* Choose some circle  $K(\varrho, \delta)$  intersecting  $\Gamma_\varepsilon$ . Assume on the contrary that  $\#(K(\varrho, \delta) \cap \Gamma_\varepsilon) > n$ . Take any  $N = n + 1$  cv from this intersection and enumerate them  $\{\varrho_j\}_{j=1}^N$ . They are distinct because we chose  $\Gamma$  not counting multiplicities. Then the rows of the matrix (7.13) are linearly dependent. Therefore for appropriate coefficients  $c_j$  in (7.11) we have

$$a_{0l} = 0, \quad l = 0, \dots, n-1. \quad (7.14)$$

Observe that for any  $\mu$  in the larger circle  $D(\varrho, \delta_1)$  max and min of the ratio  $|\mu|/|\varrho|$  are attained when  $|\mu| = |\varrho| \pm \delta_1 |\Im \varrho|$ . Therefore  $|\mu|/|\varrho| \in [1 - \delta_1, 1 + \delta_1]$  whence

$$|\varrho_j| \asymp |\varrho|, \quad j = 1, \dots, N. \quad (7.15)$$

Normalize  $c_j$

$$\sum_{j=1}^N |c_j|^2 = 1. \quad (7.16)$$

Then from (7.11), (7.16) and (7.3) stems an estimate

$$|a_{ml}|^2 \leq \sum_{j=1}^N |d_{jl}|^2 \cdot |\delta_1 \cdot \Im \varrho|^{2m} \leq \sum_{j=1}^N |d_{jl}|^2 \cdot \delta_1^{2m} \cdot |\varrho|^{2m}. \quad (7.17)$$

Substituting (7.17) into (7.12) and taking into account (7.14), (7.15) and (7.7), we obtain

$$\begin{aligned} \|u\|^2 &\leq C \cdot \sum_{l=0}^{n-1} \sum_{m=1}^{N-1} \sum_{j=1}^N |d_{jl}|^2 \cdot \delta_1^{2m} \cdot \frac{|\varrho|^{2m}}{|\varrho|^{2m+1}} \\ &\leq C \cdot \delta_1^2 \cdot \sum_{l=0}^{n-1} \sum_{j=1}^N |d_{jl}|^2 \cdot \frac{1}{|\varrho|} \\ &\leq C \cdot \delta_1^2 \cdot \sum_{j=1}^N \|u_j\|^2 = (n+1)C\delta_1^2. \end{aligned} \quad (7.18)$$

The latter contradicts (7.9), (7.16) provided  $\delta_1$  (from (7.3)) is sufficiently small.  $\square$

## 8. (UB) CONJECTURE.

For the sake of definiteness let  $\nu = 0$ . The case  $\nu = 1$  may be considered similarly. Define

$$\mathcal{D} = S_0(\varepsilon) \cap \{r \leq |\varrho| \leq r + \delta r\}.$$

Recall that the boundary of  $S_0(\varepsilon)$  are the rays  $\arg \varrho = \frac{\pi}{2n} \pm \varepsilon$ . Set  $\omega = (2\varepsilon)/M$ . We will choose the integer  $M$  later. Dissect  $S_0(\varepsilon)$  by  $M$  rays

$$T_k = \{\arg \varrho = k\omega + \frac{\pi}{2n} - \varepsilon\}, \quad k = 0, \dots, M.$$

They divide  $\mathcal{D}$  into  $M$  quadrilaterals  $\Pi_k$ ,  $k = 0, \dots, M - 1$ .

## 8.1. Quadrilaterals.

**Lemma 8.1.**  $\text{diam} \Pi \leq 2\delta r$ .

*Proof.* Since  $\Pi_k$  lies in an angle of the opening  $\omega$  and has vertexes

$$\begin{aligned} V_1 &= r \cdot \exp(i\theta), V_2 = r \cdot \exp(i(\theta + \omega)), V_3 = (r + \delta r) \cdot \exp(i(\theta + \omega)), \\ V_4 &= (r + \delta r) \cdot \exp(i\theta), \quad \theta = \frac{\pi}{2n} - \varepsilon + \omega k, \end{aligned}$$

then its diameter coincides with  $|V_3 - V_1| = |V_4 - V_2|$ . Denote  $\widehat{V_1 V_2}$  the arc between the points  $V_1, V_2$ . Let module  $|\cdot|$  stand also for the arc's length. Then

$$|V_3 - V_1| \leq |V_3 - V_2| + |\widehat{V_1 V_2}| \leq \delta r + \omega r \leq 2\delta r,$$

if we take  $M \geq \frac{2\varepsilon}{\delta}$ , say  $M = \text{entier}(\frac{2\varepsilon}{\delta}) + 1$ . □

Let  $Q$  be the center of the interval  $[V_1, V_3]$ . Then the circle  $D(Q, \delta_2)$  contains quadrilateral  $\Pi_k$  for appropriate  $\delta_2$ . Namely, it is enough if its radius  $r_Q \geq 2\delta r$ . But

$$r_Q = \delta_2 \cdot \Im Q \geq \delta_2 \cdot r \sin(\frac{\pi}{2n} - \varepsilon).$$

So we can take  $\delta_2 = 2\delta / \sin(\frac{\pi}{2n} - \varepsilon)$ .

**Lemma 8.2.** *Number of cv in  $\mathcal{D} < \frac{n}{\delta}$ .*

*Proof.* In angular direction  $\mathcal{D}$  is covered by  $M$  angles of opening  $\omega$ . An intersection of  $\mathcal{D}$  with each angle lies in  $D(Q, \delta_2)$ . Observe that in turn this circle is contained in  $K(Q, \delta_2)$ , see (7.2). Choose  $\delta_2$  as is needed in lemma 7.6. Then, according to this lemma  $K(Q, \delta_2)$  contains no more than  $n$  cv from  $\Gamma_\varepsilon$ . Hence, assuming  $2\varepsilon + \delta < 1$

$$\#(\Gamma \cap \mathcal{D}) \leq Mn \leq (\frac{2\varepsilon}{\delta} + 1) \cdot n < \frac{n}{\delta}.$$

□

8.2. Areas' estimates. Reenumerate the cv in  $\mathcal{D}$ :

$$\varrho_1, \dots, \varrho_N, \quad N \leq N_0 := \text{entier}(\frac{n}{\delta}) + 1$$

and set

$$K = \bigcup_{j=1}^N D(\varrho_j, \frac{\delta}{N_0}).$$

**Lemma 8.3.** *A relative area's estimate holds*

$$|K|/|\mathcal{D}| \leq C\delta^2/\varepsilon. \tag{8.1}$$

*Proof.* From one hand the area of  $\mathcal{D}$  is subject to an estimate

$$|\mathcal{D}| = \varepsilon \cdot (r^2(1 + \delta)^2 - r^2) > \varepsilon \cdot 2r^2\delta.$$

From the other hand the area of  $K$  admits an upper estimate:

$$\begin{aligned} |K| &\leq \sum_{j=1}^N |D(\varrho_j, \frac{\delta}{N_0})| = \sum_{j=1}^N \pi \cdot \left| \frac{\delta}{N_0} \Im \varrho_j \right|^2 \\ &\leq N \pi \frac{\delta^2}{N_0^2} \cdot \max |\varrho_j|^2 \leq N \pi \frac{\delta^2}{N_0^2} \cdot (r(1 + \delta))^2 \\ &\leq \pi \delta^2 (1 + \delta)^2 r^2 / N_0 \end{aligned}$$

whence

$$|K|/|\mathcal{D}| \leq \frac{\pi \delta^2 (1 + \delta)^2 r^2 / N_0}{\varepsilon \cdot 2\delta r^2} \leq C\delta / (N_0 \varepsilon) \leq C\delta \frac{\delta}{n\varepsilon} = C\delta^2 / \varepsilon.$$

□

We have also to take into account the area of the circles  $K(\varrho_j, \delta)$ , intersecting  $\mathcal{D}$ , such that their *centers*  $\varrho_j$  lie outside  $S_0(\varepsilon)$ . Replacing them with greater ones  $D(\varrho_j, \delta_1)$ , we see that their area attains maximum if their *centers* lie on the boundary rays of  $S_0(\varepsilon)$ . Consider for instance one of them, namely the ray  $Ray_1 = \{\arg \varrho = \frac{\pi}{2n} + \varepsilon\}$ . Then the corresponding intersections are inside the angle

$$Ang_1 = \left\{ \frac{\pi}{2n} + \varepsilon - \nu \leq \arg z \leq \frac{\pi}{2n} + \varepsilon \right\}.$$

**Lemma 8.4.** *The opening  $\nu \leq c\delta_1$ .*

*Proof.* Take  $\varrho \in Ray_1$  and draw a circle  $D(\varrho, \delta_1)$ . Clearly  $Ray_1$  is its tangent at some point  $A$ . So the radius  $\overrightarrow{\varrho A}$  is perpendicular to the  $Ray_1$  at  $A$ . Thus

$$\delta_1 |\Im \varrho| = |\overrightarrow{\varrho A}| = |\varrho| \cdot \sin \nu$$

whence

$$\sin \nu = \delta_1 \frac{|\Im \varrho|}{|\varrho|} = \delta_1 \sin\left(\frac{\pi}{2n} + \varepsilon\right)$$

and we are done since  $\sin \nu \geq \frac{2}{\pi} \nu$ ,  $\nu \in (0, \pi/2)$ . □

The intersection with circles  $D(\varrho, \delta_1)$  near the other boundary ray of  $S_0(\varepsilon)$  is estimated along the same lines and is contained within the angle

$$Ang_0 = \left\{ \frac{\pi}{2n} - \varepsilon \leq \arg z \leq \frac{\pi}{2n} - \varepsilon + \nu \right\}.$$

**8.3. Notemptiness of good domains.** Set  $Ang = Ang_0 \cup Ang_1$ .

**Lemma 8.5.** *The good domain  $\mathcal{D} \setminus K$  is not empty.*

*Proof.* First note that

$$|\mathcal{D} \cap Ang|/|\mathcal{D}| \leq \frac{2\nu}{2\varepsilon} \leq \frac{c\delta_1}{\varepsilon}. \quad (8.2)$$

Summing up the rhs of inequalities (8.1)-(8.2) we get

$$C\delta^2/\varepsilon + c\delta_1/\varepsilon = C\delta^2/\varepsilon + c \frac{2\delta}{(1-\delta)\varepsilon}.$$

Since  $\varepsilon$  is fixed, this expression can be made as small as needed for sufficiently small  $\delta$ . □

**8.4. Completion of the proof.** At last, setting  $r_m = (1 + \delta)^m$ ,  $m = 1, 2, \dots$ , we get a sequence of domains  $\mathcal{D}_m$  and thus a sequence of points  $\tau_m \in \mathcal{D}_m \subset S_\nu(\varepsilon, \delta)$ , tending to infinity. According to lemma 7.3 the resolvent obeys the estimate (4.1) with  $\varrho = \tau_m$ . It suffices to apply theorem 4.3 and theorem 4.1 is proved.

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